COUNTING IN DOZENS

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Our common number system is decimal - based on ten. The dozen system uses twelve as the base, which is written 10, and is called do. for dozen. The quantity one gross is written 100, and is called gro. 1000 is called xo, representing the meg-gross, or great-gross.

In our customary counting, the places in our numbers represent successive powers of ten; that is, in 365, the 5 applies to units, the 6 applies to tens, and the 3 applies to tens-of-tens, or hundreds. Place value is even more important in dozenal counting. For example, 265 represents 2 units, 6 dozen, and 5 dozen-dozen, or gross. This number would be called 2 grodo 6 do 5, and by a coincidence, represents the same quantity normally expressed as 365.

Place value is the whole key to dozenal arithmetic. Observe the following additions, remembering that we add up to a dozen before carrying one.

<table>
<thead>
<tr>
<th>94</th>
<th>136</th>
<th>Five ft. nine in.</th>
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<tbody>
<tr>
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<td>694</td>
<td>Three ft. two in.</td>
</tr>
<tr>
<td>96</td>
<td>522</td>
<td>Two ft. eight in.</td>
</tr>
<tr>
<td>182</td>
<td>1000</td>
<td>Eleven ft. seven in.</td>
</tr>
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</table>

You will not have to learn the dozenal multiplication tables since you already know the 12-times table. Mentally convert the quantities into dozens, and set them down. For example, 7 times 9 is 63, which is 5 dozen and 3; so set down 63. Using this “which is” step, you will be able to multiply and divide dozenal numbers without referring to the dozenal multiplication table.

Conversion of small quantities is obvious. By simple inspection, if you are 35 years old, dozenally you are only 32, which is two dozen and eleven. For larger numbers, keep dividing by 12, and the successive remainders are the desired dozenal numbers.

You may also be converted to decimal numbers by setting down the units figure, adding to it 12 times the second figure, plus 12^2 (or 144) times the third figure, plus 12^3 (or 1728) times the fourth figure, and so on as far as needed. Or, to use a method corresponding to the illustration, keep dividing by 12, and the successive remainders are the desired decimal number.

Fractions may be similarly converted by using successive multiplications, instead of divisions, by 12 or 10.

<table>
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</tr>
</tbody>
</table>

THE DUODECIMAL SOCIETY OF AMERICA

20 Carlton Place ~ ~ ~ ~ ~ Staten Island 4, N.Y.
The Duodecimal Bulletin

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Mathematicians became conscious of a new dimension in symbolism, and the fundamental concepts of number were re-examined. Man awoke to the fact that different number bases could be used, and Simon Stevin stated in 1585 that the duodecimal base was to be preferred to the decimal.

The new Arabic notation accommodated mathematical statement better, and facilitated ideation. All thinking accelerated when released from the drag of the cumbrous Roman notation.

The parallel seems tenable. The notation of the dozen base accommodates mathematical statement better, and facilitates ideation. It, too, is a step forward in numerical symbolism. The factorable base embodies a concurrent analysis and definition of numbers that stimulates classification and generalization. Yet this is accomplished by such simple means that students in the primary grades easily learn to perform computations in duodecimals, and can tell why they are better. Literally, the decimal base is unsatisfactory because it has “not-enough-factors.”

Then, shouldn't we change? No! No change should be made, and we urge no change. All the world uses decimals. But people of understanding should learn to use duodecimals to facilitate their thinking, and to ease the valutative processes of their minds. Duodecimals should be man's second mathematical language. They should be taught in all the schools. In any operation, that base should be used which is most advantageous, and best suited to the work involved. We expect that duodecimals will progressively earn their way into
The Duodecimal Bulletin

AN EXPOSITIONAL EXPRESSION FOR MUSIC

by Trenchard More, Jr.

INTRODUCTION Let me first congratulate Velizar Godjevatz on his new musical notation. After taking ten years of piano lessons, I am still pitifully slow at reading, and perhaps now I will never bother to learn that impossible collection of old conflicting notations. The new stave should bring the pleasure of rapid musical reading within the scope of those who are unable to practice for hours a day. I have tried writing music solely in dozional numbers, but this presents too much detail for the eye to grasp. Because of its greater simplicity, the Notation Godjevatz lies closer to beauty than the old musical script.

DIFFICULTIES However, the dozional numbers representing Godjevatz’s “Absolute Pitch” remain mathematically arbitrary, as well as the “Audition Range”, most existing pitch standards, and most existing notations. Although the idea of numbering notes is a good one, the present duodecimal musical numbers are a logical sequence of symbols rather than a powerful mathematical tool. The lettering of notes appears to be based on habit rather than necessity, both in the new and old notations, and when these letters evolve into such forms as $\#$ or $\#$, the spirit of simplicity revolts. In the field of theoretical music there are apparently as many different alphabetic notations as there are experimenters.

OBJECTIVE If possible, we must adopt in our duodecimal simplification of science a system of music satisfactory to the artist, physicist, and mathematician. This system should be able to contain and encourage future developments in music without having to be discarded for a different method. It should be thoroughly grounded, unified, and flexible.

DEFINITIONS Sound is divided into the two categories of noise and musical tone, which are distinguished by periodicity. Noise has no period; that is, it does not repeat itself in constant intervals of time. A musical tone has force (the amplitude of the sonorous body), pitch (the rate of periodicity), and quality (how the sonorous body vibrates within each period). We are concerned only with absolute pitch (pitch standards and pitches of musical instruments) and relative pitch (musical intervals: octave, fourth, fifth, etc.). We define a cycle as a period, and frequency as the rate of periodicity or the number of cycles per unit time. A musical interval is a ratio of
the frequencies of two notes in cycles per unit time. A chord
is composed of two or more intervals, or three or more musical
tones.

**Basic Idea** It is necessary to select a suitable musical inter-
val as the standard for all other intervals. The ear has the
peculiar property that it enjoys simple numerical ratios; such
as 1:2, 2:3, 3:5, etc. Of these, the simplest, and most used
interval in music is the frequency ratio of 1:2, or the octave.
The choice of the octave as the standard interval is the only
arbitrary assumption in this system. From it, the notation
and all else is derived mathematically. Of course there are
many discriminations to be made, but these will be based on
our first assumption. The choice of a unit of time will be
mentioned later.

The curve of pitch must double itself in every progressive
octave. Therefore,

\[ y = 2^x \]

(1)

where \( y \) is the frequency in cycles per unit time (pitch), and
\( x \) is the number of octaves (intervals of 1:2). Equation (1)
represents an exponential curve which intersects the y (ordinate)
axis at the point \((x = 0, y = 1)\). The curve lies above the
x axis, and approaches it asymptotically to the left. We need
only that portion of the curve which lies in the first quad-
 rant, upper right. At present, an arbitrary point on this
curve is used for a standard, which has the decimal coordinates
(8.781, 440). This standard leads to a complicated formula for
the calculation of the American Equally Tempered Chromatic
Scale, to which most pianos are tuned.

\[ y = 2 \log_2 440 - \frac{N}{12} \]

(2) dec

where \( y \) equals the frequency in cycles per second, and \( N \) equals
the number of semitones above or below A, 440. For the semi-
tones below 440, \( N \) becomes negative.

It seems better to avoid arbitrary constants and to retain
the simple form of equation (1) by letting our standard point
on the curve be the \( y \) intercept \((0, 1)\); (grounded). Then all
frequencies, absolute and relative, may be represented directly
by exponents of 2; (unified). Instead of multiplying and divid-
ing frequencies, we add and subtract their exponents. We will
abolish both the alphabetic notation for absolute pitch, and
the Roman numeral notation for relative pitch by using their
exponents only. Then any absolute pitch, and any musical inter-
val may be represented to any desired degree of accuracy by an
exponent of 2; (flexible). Here there are no constants, stand-
dards, and symbolic notations to hamper accuracy and expres-
sion. There are no arbitrary and complicated conversion methods to
apply between absolute and relative pitch, notation and actual frequency, or one notation and another. The exponential expression for music is built as closely as possible about the curve $y = 2^x$. If those who advance further in the field do the same, unification may be preserved. I hope that this curve is basic and flexible enough to accommodate the now unforeseen advances in music.

**INTERVALS** Since scales are based on intervals, the interval is mentioned first. We wish to find an exponential representation for the frequency ratio of $v : w$ cycles per unit time. This is given by the equation

$$\frac{\log v - \log w}{\log 2} = \frac{v}{w} > w > 0$$

where the log may have any convenient base, the most suitable for this paper being do (one dozen). Then all intervals of the form $v : w$ will be represented by the numerical equivalent of $\frac{\log v - \log w}{\log 2}$ instead of arbitrary names, numerals, or letters. The table “Names and Ratios of Intervals” has been calculated for the more prominent musical intervals. Try adding the pure fourth and fifth together, or the major third and the minor sixth, or the minor third and the major sixth, or the major second and minor subdominant seventh, and you will note the ease with which these intervals may be handled.

The “Interval Chart” may be of aid to the experimenter. To find the interval for the ratio of 3 : 8 or 8 : 3, trace the diagonal leading from 3 to the point where it intersects the diagonal leading from 8. Immediately to the right of the arrowhead formed by these two diagonals is the answer 1.4292. Every other diagonal has been omitted to ease the eye, so that every whole number on the left has two diagonals leading from it; one blank; and one drawn. Of course this table may be extended, but nearly all of the useful intervals are included within the combinations of the first dozen (14) digits. The experimenter may find some very pretty “runs” in this triangle by playing each interval of a diagonal in succession. There are also several arithmetical properties in the table which increase its intrigue, and facilitate its computation.

**SCALES** A scale is a series of absolute pitches which double their frequencies in each progressive octave. Thus each octave of notes is a duplicate of the next, except at a different level of pitch. The notation for absolute pitch is given directly by equation (1), and is more easily calculated by equation (4).
\[ y = 2^x \]  \hspace{1cm} (1)

\[ x = \log_2 y = \frac{\log y}{\log 2} \]  \hspace{1cm} (4)

To find the notation for 194.00 cycles per unit time, substitute 194 for \( y \) in equation (4), and receive the answer \( x = 8.0 \). Then \( 2^8 = 194 \), and in this exponential expression for music the absolute pitch of 194.00 cycles per unit time will always be written as 8.0 instead of C\(_4\#\) or 194 or B\(_3\)\#. The octaves above this note would be written as 9.0, \( \#\# \), etc., and the frequencies of these notes would be 368.00, 714.00, etc. A fifth above \( \#\# \) 8.0 would be written as \( \#\#\#\# \), and a tempered fifth above \( \#\#\#\# \) 8.0 would be \( \#\#\#\#\# \). Thus by adding intervals to pitches, we get new pitches; all in a unified notation.

To return to intervals for a moment, suppose we have the ratio 4:3, this is the same as 1.4:1. Calculating our notation for 1.4 cycles... 0.4292, and subtracting our notation for 1 cycle... 0.0000, we have 0.4292, which expresses both the interval of 4:3, and the absolute pitch of 1.4 cycles. Then intervals and pitches may be thought of as the same, except that the useful intervals will range from 0.0 to 4.0, and the useful pitches will range from 4.0 upward; that is, if we use a unit of time equal to the second. More about time later.

The “just” or “scientific” major scale is composed of the following ratios: 9:8, 5:4, 4:3, 3:2, 5:3, 13:8, and 2:1, which complete one octave. The intervals of these ratios determined from the Interval Chart establish the initial octave ranging from 0.0 to 1.0. To find the notation for the eighth octave, add 8.0 to each interval of the initial octave. Then if it is necessary to find the frequency directly from the notation, multiply the notation (exponent) by \( \log 2 \), and take the antilogarithm of the product (see equa. 4). With the use of the log log scales on the proposed duodecimal slide rule, a large portion of the curve \( y = 2^x \) could be laid out in infinite graduation with but one setting of the slide, thus making the calculations between frequency and notation a matter of tabulation.

**Temperament** Scales are the result of compromises between manipulation and harmony. If we were to play in only one key, matters would be simple, but instead we prefer to play in two tonal modes, and in as many keys as there are notes in the resulting octave; all of this in accurate just intonation. Each key would require new notes, which would require new keys, etc., perhaps making the keyboard octave several feet wider than a grand piano. Obviously temperament is necessary; a subject which has been discussed since the time of the Greeks, who had seven tonal modes instead of two.
The interval of next importance to the octave is the fifth.
If we add a dozen fifths together, we have an interval of
7.0299, which differs from an interval of 7 octaves by an error
of .0299. Then distributing this error evenly among the dozen
fifths, we have the tempered fifth of .7000, differing from
the true fifth by the error .00299. This then is an argument
for our modern tempered scale, for an interval of .00299 (two
decimal cents) is scarcely perceptible to the most trained ear.
Equal temperament may be thought of as the reduction of every
interval to the nearest dozenth place. The error column in
"Just and Tempered Scales" shows that just thirds and sixths
suffer most from temperament. Now add 1.0, 2.0, 3.0, etc., to
each interval in the tempered octave, and you have the notation
for the Duodecimal Equally Tempered Chromatic Scale, a notation
similar to the one which Godjevatz suggested.

### JUST AND TEMPERED SCALES

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<th>others</th>
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<td>9:5</td>
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<td>.0000</td>
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</table>

### TIME
Until now, the customary phrase 'cycles per second' has
been carefully substituted by 'cycles per unit time'. This was
done to emphasize that only one arbitrary assumption was made
(the choice of the octave, all else derived mathematically), and
that the curve $y = 2^x$ is independent of time. You may seat your-
self at the piano now, without regard to absolute pitch or time,
and play the diagonals of the "Interval Chart" by taking each
interval to the nearest dozenth place. You may use the tables
"Names and Ratios of Intervals", "Just and Tempered Scales",

### DUODECIMAL EQUALLY TEMPERED CHROMATIC SCALE
graciously checked by Mr. George S. Terry

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</table>

10.0 2454.0
"The Curve \( y = 2^x \)" and the "Interval Chart" without once considering a unit of time. Instrument designers, piano tuners, and physicists are the ones who worry about absolute pitch. The most important parts of artistic music are the combinations and sequences of intervals and relative pitches. The whole theory and notation of intervals is complete with but one assumption, that of the octave.

Now to apply our notation to absolute pitch, we must make a second arbitrary assumption and choose a unit of time, thereby placing the limits of audible sound upon the curve. If we use a time unit of one second, the interval of 0.04623 (equals \( \log 308 - 8.9 \) duo) may be added to each exponent in the \( x \) (or notation) column of the "Duodecimal Equally Tempered Chromatic Scale" to find the notation for the Equally Tempered Chromatic Scale of American Standard Pitch A 440 (dec). 308 ( duo). Then our notation for 308 (A 440) cycles per second would be 8.94623. However, if a time unit other than the second is used, we will find it necessary to convert from one time system to another.

Let there be two systems in our notation, one based on time unit \( S \), the other based on time unit \( T \), where \( (T) (K) = S \), \( K \) is a known constant factor between one unit time \( S \) and one unit time \( T \). Suppose we have an absolute pitch (call it \( A \)) which is sounded by \( U \) cycles per time unit \( T \). Then \( A \) will also be sounded by a frequency of \( (U)(K) \) cycles per time unit \( S \). Now the notation for \( U \) cycles per \( T \) is \( \log U \), and the notation for \( U \) cycles per \( S \) is also \( \log U \), but the absolute pitches of these two notations are different due to the difference in the time units. It seems convenient to be able to convert the notation for absolute pitch \( A \) in time system \( T \) to the notation for the same pitch in time system \( S \), by the addition or subtraction of a constant interval. This is handled by equation (5)

\[
\log U \log 2 + \log K \log 2 = \log (UK) \log 2; \quad \text{where } (TK) = S \quad (5)
\]

Let time unit \( S \) be one second, and let time unit \( T \) be \( \frac{1}{2000} \) (duo) part of an hour, call it one threeadvic (30 Vics).

There are 2100 seconds in one hour, and 2000 threedovics in one hour. Therefore \( T \left( \frac{20}{21} \right) = S \). Then suppose we have 194 cycles per threedovic; our notation for this is 8.0. What should the notation be for the same pitch (194 cycles per threedovic) in cycles per second? Substitute the above values in equation (5). \(-0.085 \, 928\) is the conversion interval for all frequencies.

\[
\left\{ \log 194 \right\} = 8.0 \quad \left[ \begin{array}{c} \log 20 - \log 21 \end{array} \right] \quad \log 2 \quad = -0.085 \, 928 \quad = 7.23629 \quad \text{(ans.)}
\]

It seems better to stick to the second, for a while at least, because time is already sufficiently dozenal for practical purposes, and to abolish it would raise a psychological barrier inquiring minds of laymen. The dozenal quality of time is one of the few bridges which many may cross to duodecimals. It is a "foot in the door" so to speak. However, the choice of a unit of time is left to the reader.

However, we should aim for a duodecimal division of the day, just as we have the duodecimal division of the circle. The choice of a suitable fraction of the day as the unit of time for this musical notation is left to the reader.

**SUMMARY** Someone might suggest that it would be simpler to use the frequency numbers themselves, rather than all these logarithms and exponents. It would not be for three reasons. First, musical frequencies are usually handled by multiplication and division, whereas their logs (on base two calculated with the aid of base dozen), are added and subtracted. Second, interval distances between frequencies do not remain constant for one sort of sound sensation (octave, fifth), whereas the interval distances between the logarithms of the frequencies of a given ratio remain constant at every level of pitch (see "The Curve \( y = 2^x \)). Third, the Equally Tempered Chromatic Scale cannot be expressed in simple frequency numbers, whereas their logarithms may be expressed with two digit numbers for the first dozen octaves.

It might be well to mention a few advantages of our notation. It should be understood that this article was designed to simplify the notation of theoretical music. The musician will be concerned primarily with Godjevart's Stave, the two digit notation of the Duodecimal Equally Tempered Chromatic Scale, and the tempered intervals .70, .50, 1.0, etc.
The complete solution of that interesting theory of numbers problem \( A^2 + B^2 = N = (P_1^A)(P_2^B)(P_3^C) \ldots \) may be found by a simple trigonometric device. Since the method to be reported here can be derived from the solution obtained through the expansion of complex numbers by D. Chelini, (see Dickson's "History of the Theory of Numbers", vol. II, pg. 239), no derivation or proofs will be given. A description of the method of solution will serve to outline the interesting relationships involved.

If \( P \) is an odd prime of the form \( 40p + 1 \) or \( 10p + 5 \) (using dozenal notation,) its representation as the sum of two squares is said to be unique, that is, \( P = a^2 + b^2 \), in one way and one way only where \( (a, b) \) are integers with no common factor greater than 1. Now every prime, \( P \), which can be so represented, can also be represented by one and only one angle, \( \gamma \), the tangent of which is, \( \tan \gamma = \frac{b}{a} = \frac{B}{A} \).

Substituting in \( A^2 + B^2 = N \), \( P = a^2 + b^2 \).

Now if \( N = (P_1^A)(P_2^B) \) and \( \gamma_1 \) and \( \gamma_2 \) are the angles corresponding to \( P_1 \) and \( P_2 \) and their tangents are, \( \tan \gamma_1 = \frac{b}{a} \), \( \tan \gamma_2 = \frac{c}{d} \), first we determine the tangent of the sum of the angles, -

\[
\tan (\gamma_1 + \gamma_2) = \frac{(b/a) + (c/d)}{1 - (b/a)(c/d)} = \frac{B_1}{A_1} = \frac{bc + ad}{ac - bd}
\]

next we determine the tangent of the difference between the angles, -

\[
\tan (\gamma_1 - \gamma_2) = \frac{(b/a) - (c/d)}{1 + (b/a)(c/d)} = \frac{B_2}{A_2} = \frac{bc - ad}{ac + bd}
\]

Now we have \( A_1^2 + B_1^2 = A_2^2 + B_2^2 = N = (P_1^A)(P_2^B) \), and substituting we obtain:

\[
(ac - bd)^2 + (bc + ad)^2 = (ac + bd)^2 + (bc - ad)^2 = (a^2 + b^2)(c^2 + d^2)
\]

Thus by simply finding the tangents of the sum and difference of two angles, we have arrived at the familiar four parameter identity for two sums of two squares. If one of the two
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Numerical example of the six parameter solution:
Let \( a = 2; \ b = 1; \ c = 3; \ d = 2; \ e = 4; \ f = 1; \) and we obtain

(Base X)
\[ 33^2 + 4^2 = 32^2 + 9^2 = 31^2 + (-12)^2 = 24^2 + 23^2 = 1105 = 5.13.17 \]

(Base XII)
\[ 29^2 + 4^2 = 28^2 + 9^2 = 27^2 + (-10)^2 = 20^2 + 1\ell^2 = 781 = 5.11.15 \]

So far we have only considered composites made up of the first powers of primes. We should now consider the representations of higher powers of primes as the sum of two squares. Let the angle, \( y \), the tangent of which is \( \tan y = \frac{b}{a} \), represent a prime, \( P \). We determine,

\[
\tan (y + y) = \frac{(b/a) + (b/a)}{1 - (b/a)(b/a)} = \frac{B_1}{A_1} = \frac{2ab}{a^2 - b^2}
\]
\[
\tan (y - y) = \frac{(b/a) - (b/a)}{1 + (b/a)(b/a)} = \frac{B_2}{A_2} = \frac{0}{a^2 + b^2}
\]

Now substituting in, \( A_1^2 + B_1^2 = A_2^2 + B_2^2 = P^2 \)
we obtain\( (a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2 + o^2 = P^2 \)

This last expression is obviously our familiar two parameter solution for the Pythagorean Triangle. Nothing new, but the generators, \( (a, b) \), instead of being figures pulled out of the air have an added dignity. They define the tangent of an angle, an angle that is exactly half of one of the angles of the triangle which they generate.

A complete investigation of the trigonometric meaning of these generators is beyond the scope of this introductory paper but we should note several interesting relationships. If \( (a, b) \) are relatively prime, one even and the other odd, the triangle which they generate will be "primitive". Obviously all multiples of this triangle will have the same angles even though the generators appear to change. Now if we multiply both numerator and denominator of \( \tan y \) by \( \sqrt{k} \), thus,

\[
\tan y = \frac{b}{a} = \frac{b\sqrt{k}}{a\sqrt{k}}
\]

and let \( k = 1, 2, 3, 4, \ldots \), we obtain all multiples of the "primitive" triangle. Now we also know that there are two acute angles to the triangle and if it is generated by the tangent of half of one angle, then what of the other angle? Let

and substituting in, \( A_1^2 + B_1^2 = A_2^2 + B_2^2 = P^2 \)
we obtain\( (a^2 + b^2)^2 + (2ab)^2 = (a^2 - b^2)^2 + o^2 = P^2 \)

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\]

and let \( k = 1, 2, 3, 4, \ldots \), we obtain all multiples of the "primitive" triangle. Now we also know that there are two acute angles to the triangle and if it is generated by the tangent of half of one angle, then what of the other angle? Let
the two half angles be $y_A$ and $y_B$. We know the sum of the half angles is half of one right angle, that is, is the angle whose tangent is $1$. Thus if

$$\tan y_A = \frac{b}{2},$$

we may write

$$\tan y_B = \frac{1 - b/a}{1 + b/a} = \frac{a - b}{a + b}$$

and as we have shown earlier, if $a^2 + b^2 = P$, then $(a - b)^2 + (a + b)^2 = 2P$, and using $(a + b, a - b)$ as generators we obtain the two-multiple of the primitive triangle generated by $(a, b)$. From this relationship we find that all angles with rational tangents occur in pairs where $\tan y_A = \frac{b}{a}$, with either $a$ or $b$ even, the other odd, and $\tan y_B = \frac{a - b}{a + b}$, with both numerator and denominator odd.

Thus every angle with a rational tangent is a member of such a pair, and every such pair determines a primitive Pythagorean triangle and all of its multiples. Note that the triangle is actually determined by the pair although only one angle is used. Before leaving this subject we might also note, since all trigonometric functions of a Pythagorean triangle are rational, that, if $y$ is any angle with a rational tangent, then all functions of $(2y)$ are rational.

Proceeding to the next step, $A^2 + B^2 = P^2$, we determine,

$$\tan (2y + y) = \frac{B}{A} = \frac{3a^2b - b^2}{a^3 - 3ab^2}$$

$$\tan (2y - y) = \frac{B}{A} = \frac{a^2b + b^3}{a^3 + 3ab^2}$$

Now substituting in $A^2 + B^2 = A_1^2 + B_1^2 = N = P^2$ we obtain

$$(a^2 - 3ab^2)^2 + (3a^2b - b^3)^2 = (a^3 + ab^2)^2 + (a^2b + b^3)^2 = (a^2 + b^2)^2 = P^2$$

For a numerical example for this last case, let $a = 2; b = 1$ and we obtain (Base X), $11^2 + 2^2 = 10^2 + 5^2 = 5^2 = 125$

or for Base XII,

$$2^2 + 2^2 = 5^2 + 5^2 = 5^2 = 5^2$$

This method may be continued to obtain two parameter solutions for the partitioning of any power of $P$ into two squares.

We may state the general solution for $A^2 + B^2 = P^r$ as follows:

Determine all values of $\tan Z = \frac{B}{A}$

where $Z = (r - n) \gamma - ny$

and $y$ is the angle which represents the prime $P$.

and $n = 0, 1, 2, \ldots, r$, if all representations are required, or $N \leq (r/2)$ if difference in sign of quantities to be squared can be ignored.

After obtaining all values of $\tan Z$, we take the sum of the squares of the numerator and denominator of each value and this sum forms one member of the identity, which will have as many members as there are values of $\tan Z$. Care must be taken not to simplify the values of the tangent since it will be noted that when $n$ has any value other than zero, the numerator and denominator can both be divided by $P$.

The results for $P_1, P_2, P_3, \ldots$, etc., may be combined exactly as for composites of the first powers by combining the angles $Z_1, Z_2, Z_3, \ldots$ etc. Thus we may solve, for example, $A^2 + B^2 = N = P_1^2P_2^2$. Let $P_1 = a^2 + b^2$, $\tan y_1 = \frac{b}{a}$, and $P_2 = c^2 + d^2$, $\tan y_2 = \frac{d}{c}$, then we find that

$$A_1^2 + B_1^2 = A_2^2 + B_2^2 = A_3^2 + B_3^2 = A_4^2 + B_4^2 = A_5^2 + B_5^2 =$$

$$A_6^2 + B_6^2 = (a^2 + b^2)^3(c^2 + d^2)^2$$

is solved by,$$
A_1 = a^2c^2 - 3ab^2c^2 - a^3d^2 - 3a^2bd^2 - 6a^2bcd + 2b^2cd
B_1 = 3a^2bc^2 - b^3c^2 - 3a^2bd^2 + b^2d^2 + 2a^3cd - 6ab^2cd
A_2 = a^2c^2 - 3ab^2c^2 - a^3d^2 - 3a^2bd^2 + 6a^2bcd - 2b^2cd
B_2 = 3a^2bc^2 - b^3c^2 - 3a^2bd^2 + b^2d^2 - 2a^3cd + 6ab^2cd
A_3 = a^2c^2 - 3ab^2c^2 + a^3d^2 - 3a^2bd^2
B_3 = 3a^2bc^2 - b^3c^2 + 3a^2bd^2 - b^2d^2$$
$A_4 = a^2c^2 + ab^2c^2 - a^2d^2 - ab^2d^2 - 2a^2bcd - 2b^2cd$
$B_4 = a^2be^2 + b^2e^2 - a^2bd^2 - b^2d^2 + 2a^2cd + 2ab^2cd$
$A_5 = a^2c^2 + ab^2c^2 - a^2d^2 - ab^2d^2 + 2a^2bcd + 2b^2cd$
$B_5 = a^2be^2 + b^2e^2 - a^2bd^2 - b^2d^2 - 2a^2cd - 2ab^2cd$
$A_6 = a^2c^2 + ab^2c^2 + a^2d^2 + ab^2d^2$
$B_6 = a^2be^2 + b^2e^2 + a^2bd^2 + b^2d^2$

For a numerical example, let $a = 2$; $b = 1$; $c = 3$; $d = 2$; and we have Base XI,

$(-122)^2 + 79^2 = 142^2 + 31^2 = 26^2 + 143^2 = 52^2 + 11^2 = (52)^2(11^2) = 21125$

Base XII

$(-X)^2 + 67^2 = 2X^2 + 27^2 = 22^2 + 20^2 = (22)^2 + (20)^2 = (53)^2(11^2) = 10285$

Using methods identical with the foregoing we may also solve the equation $A^2 - B^2 = N$, where every prime, $P$, may be represented by the tan $y = \frac{b}{a}$, and $y_1$, $y_2$, $y_3$, etc., are combined as usual for obtaining the hyperbolic tangent of the sum and differences of quantities. In this problem it is obvious that $P = a^2 - b^2$.

The duplication of a complete solution by other methods is of little or no importance. It may be important that we have found some interesting relationships between trigonometry and number theory that have been neglected since the days of the Babylonians who used tables of Pythagorean Triangles to measure the trigonometric functions. A thorough investigation of the trigonometry of number theory may be a fertile field for extension of our knowledge.

MATHMATICAL RECREATIONS
D. M. Brown, Editor

In the last issue we requested suggestions for new kinds of recreational material. So far the suggestion box is sadly neglected. However, several errors have been called to our attention, and we are glad to get some mail.

In the June, 1949 issue of this publication, the answer to problem 1 should be $x = 1$, $y = 6$. In problems 3 and 4, the last plus sign should be replaced by an equality sign.

Lewis C. Seelbach submitted a "Lazy" solution to the magic square given on page 15 of the last issue. The solution is

LAZY COMPUTER
0 1 2 3 4 5 6 7 8 9 X 2

There is enough information given in the problem as stated to solve it by use of algebra. We'll be glad to publish an algebraic solution.

To emphasize the value of various bases in easy solution to certain types of problems, the following problems are submitted:

1. The weight problem.

A merchant has a scale for weighing articles consisting of two balanced pans. He desires to be able to weigh articles weighing from 1 to 100 lbs. What is the most economical set of weights to use? What number-base is involved?

2. The coin problem.

The same merchant is aware that of $N$ coins of the same denomination, one is spurious, and of different weight than the others. Using the balance and the coins, how many weighings must he be permitted to make to determine the following:

a. Which coin is spurious, if he knows that the spurious coin is heavier (or lighter) than the others?

b. Which coin is spurious if he knows only that its weight is different from the others?

c. Which coin is spurious, and which is heavier, the spurious coin or a good one, if he knows only that the good and spurious coins have different weights?

d. How many weighings are permitted if $N = 3, 4, 5, 6, \ldots$ etc?

e. What number base is most convenient for the problem?
HALVING

One of the simple ways of dividing things into small parts is to successively divide them into halves or thirds, again and again, until the desired fraction is reached. Man has frequently shown his preference for arranging the scales of his standards of weight and measure to facilitate this type of subdivision. Familiar illustrations are the foot with twelve inches, and the pounds of twelve and sixteen ounces. In analyzing factors of number bases for general use, such divisibility is one of the important criteria. The following table will afford a visual comparison of the degree of facility of successive halving on different bases.

<table>
<thead>
<tr>
<th>Fractions</th>
<th>Base Two</th>
<th>Base Eight</th>
<th>Base Ten</th>
<th>Base Twelve</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>.1</td>
<td>.4</td>
<td>.5</td>
<td>.6</td>
</tr>
<tr>
<td>1/4</td>
<td>.01</td>
<td>.2</td>
<td>.25</td>
<td>.3</td>
</tr>
<tr>
<td>1/8</td>
<td>.001</td>
<td>.1</td>
<td>.125</td>
<td>.16</td>
</tr>
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<td>1/16</td>
<td>.000 1</td>
<td>.04</td>
<td>.062 5</td>
<td>.09</td>
</tr>
<tr>
<td>1/32</td>
<td>.000 01</td>
<td>.02</td>
<td>.031 25</td>
<td>.046</td>
</tr>
<tr>
<td>1/64</td>
<td>.000 001</td>
<td>.01</td>
<td>.015 625</td>
<td>.023</td>
</tr>
<tr>
<td>1/128</td>
<td>.000 000 1</td>
<td>.004</td>
<td>.007 812 5</td>
<td>.011 6</td>
</tr>
<tr>
<td>1/256</td>
<td>.000 000 01</td>
<td>.002</td>
<td>.003 906 25</td>
<td>.006 9</td>
</tr>
<tr>
<td>1/512</td>
<td>.000 000 001</td>
<td>.001</td>
<td>.001 953 125</td>
<td>.003 46</td>
</tr>
</tbody>
</table>

It is surprising that the twelve base should approach the octic base as closely as it does in the ease of accommodating this operation. The ten base requires exactly as many places in each step as does the binary base, - the twelve base half, and the octic base a third as many places.

AN ALPHABETICALCALCULATION

by Philip Haendiges

A Establishes Justifiably Of
Bright Figures Killing Problematical
Calculator Giving Long Quantities.
Duodecimally Him Mathematical
Information Notations

Saving With
Time, X-ellent
Understanding Yardsticks
Values, Zero-iferous.

PYTHAGOREAN TRIANGLES

and THEIR INSCRIBED CIRCLES

by H. C. Robert, Jr.

In the Bulletin of October 1948 (Vol. 4, No. 2, page 15), a brief discussion of Pythagorean Triangles with equal perimeters was presented. An even more neglected subject is the relationship between Pythagorean Triangles and the circles inscribed in them.

The relationship, \( A^2 + B^2 = C^2 \), is usually derived from the generators, \((m, n)\) where \( A = m^2 - n^2, B = 2mn, C = m^2 + n^2 \). In terms of these generators, the radius, \( R \), of the circle inscribed in a Pythagorean Triangle is \( R = n (m - n) \).

For \( R = 1 \), it is obvious that \( n = 1 \) and \( (m - n) = 1 \), or \( m = 2 \). These generators produce our smallest Pythagorean Triangle, the 3, 4, 5, triangle of Figure 1. Since the sides, 3, 4, 5, have no common factor, this triangle is called a "primitive" triangle. Such triangles are produced if, and only if, the generators \((m, n)\) are relatively prime and one of them is even. No other right triangle with integral sides can be circumscribed about the circle of unit radius. Obviously, a multiple of the 3, 4, 5, triangle can be circumscribed about every circle with an integral radius.

Note how the foregoing corresponds exactly with the characteristics of unity, that is, the only integral factor of unity is unity itself and unity is a factor of any integer.

Before passing on to other triangles, it is interesting to note some of the relationships between a Pythagorean Triangle and its inscribed circle:

1. The tangents from the vertices to the inscribed circle are necessarily integral.
2. The tangent of half of the vertex angles is rational. All functions of the whole angles are rational.

3. The excess of the sum of the two legs of the triangle over the hypotenuse is equal to twice the radius of the inscribed circle.

4. The area of the triangle is the product of half its perimeter and the radius of the inscribed circle. The half-perimeter can also be represented as the sum of the tangent distances from the three vertices to the inscribed circle.

Now let us investigate the case of the circle with a radius of 2. We can circumscribe two and only two right triangles about this circle:
1. \((3, 2)\) \(A = 5, B = 10, C = 11\)
2. \((3, 1)\) \(A = 8, B = 6, C = 10\)

The first of these, the 5, 10, 11, triangle is primitive and a multiple of this triangle can be circumscribed about every circle with an even radius. The second triangle is not primitive, being the 2-multiple of the 3, 4, 5, triangle for the unit circle. No other integral right triangles can be circumscribed about the circle with radius, \(R = 2\).

Now for \(R = p\), where \(p\) is any odd prime, we find that three and only three triangles may be circumscribed about such a circle, thus:
1. \((p + 1, p)\) \(A = 2p + 1, B = 2p^2 + 2p, C = 2p^2 + 2p + 1\)
2. \((p + 1, 1)\) \(A = p^2 + 2p, B = 2p + 2, C = p^2 + 2p + 2\)
3. \((2\sqrt{p}, \sqrt{p})\) \(A = 3p, B = 4p, C = 5p\)

The first two triangles are primitive, the third is the \(p\)-multiple of the 3, 4, 5, triangle for the unit circle. Actually the first of these triangles is primitive not only for prime values of \(p\) but for every integral value of \(p\), even or odd. It is the well known case for \(C = B - 1\), the solution of which is generally credited to Pythagoras. The second triangle is also a well known case, that of \(C - A = 2\), and this triangle will be primitive for all odd values of \(p\) whether prime or composite. For cases of even \(p\), this triangle will be the 2-multiple of the first triangle as may be verified by substituting \(2p\) for \(p\) in the second triangle and comparing the result with the first triangle multiplied by 2. Multiples of these two primitive triangles for each prime, (or odd composite) can be circumscribed about every circle, the radius of which is a multiple of \(p\).

If \(R = ab\), where \(a\) and \(b\) are two distinct odd primes greater than 1, we get the three same triangles, two primitive and one the \(p\)-multiple of 3, 4, 5, as given previously for a prime radius. It is only necessary to substitute \(p = ab\) in the forms given. We also get the \(a\)-multiples of the two primitive triangles belonging to the prime, \(b\), thus: \(A = 2ab + a, B = 2ab^2 + 2ab, C = 2ab^2 + 2ab + a\), and \(A = ab^2 + 2ab, B = 2ab + 2a, C = ab^2 + 2ab + 2a\) and we get the \(b\)-multiples of the two primitive triangles belonging to the prime, \(a\), thus: \(A = 2ab + b, B = 2ab^2 + 2ab, C = 2ab^2 + 2ab + b\) and \(A = a^2b + 2ab, B = 2ab + 2b, C = a^2b + 2ab + 2b\) and in addition we get two new primitive triangles, thus: \(a + b, a\) \(A = 2ab^2 + b^2, B = 2a^2 + 2b, C = 2a^2 + 2b + b^2\) \(a + b, b\) \(A = a^2 + 2ab, B = 2ab + 2b^2, C = a^2 + 2ab + 2b^2\)

Thus for an odd composite with two distinct factors, we have the same three triangles that would characterize a prime, and in addition we have six other triangles, four non-primitive and two primitive. It is obvious from the distribution of the two factors in the several members of these six additional triangles, that if any one of these triangles is known, we can find the factors of the radius \(R = ab\). Unfortunately, we know of no way of finding any one of these triangles unless we first know the factors of \(R\). Otherwise we would have a solution to the eternal problem of factorisation.

When the radius, \(R\), is even, there are two or more primitive triangles except when \(R\) is a power of 2, for which case there is only one primitive triangle. The number of non-primitive triangles is a rather complicated function of the factors of the radius, \(R\), but the number of primitive Pythagorean Triangles, \(P_T\), which can be circumscribed about any given radius, \(R\), can be simply stated, thus: For \(R = 2^n f_1 f_2 f_3 \ldots f_n\) where \(f_1, f_2, \ldots, f_n\) are \(n\) distinct odd primes greater than 1, the number of primitive triangles, \(P_T\), is: \(P_T = 2^n\). Thus when \(R = 1\) or \(R = 2^1\), we will have \(n = 0\) and \(P_T = 2^n = 1\) and when \(R = p\), \(p\) being an odd prime, \(n = 1\) and \(P_T = 2\), as has been stated previously.

Since there is at least one, and generally two or more primitive Pythagorean Triangles for every integral value of \(R\), it appears that there are many more such triangles than there are numbers. For example, there are 278 primitive Pythagorean Triangles for which the radius of the inscribed circles does not exceed 100.
All Pythagorean Triangles which may be circumscribed about a circle of radius, $R$, may be found from the factorisations of $R = k_1a_1b_1 = k_2a_2b_2 = k_3a_3b_3 = k_4a_4b_4$ where $k_1$, $k_2$, etc., either equal $1$, or are odd numbers without square factors. There will be as many circum-
scribed triangles as there are different factorisations including reversals of order of the factors $a$ and $b$. The triangles will be primitive if, and only if, $k = 1$ and $a$ and $b$ are relatively prime and $b$ is odd. The generators from which the sides may be found are obtained from these factorisations of $R$ by means of:

$$m = (a + b) \sqrt{k}$$

$$n = a \sqrt{k}$$

The resemblance between the foregoing and the method of handling perimeter problems given in the Bulletin of October 1948 is obvious.

This brief discussion does little more than scratch the surface of the neglected subject of the relationships between Pythagorean Triangles, the circles inscribed in them, the tangent distances and other items that are introduced when we consider the inscribed circles. Further investigation of the subject may be both interesting and profitable. A table giving all triangles with a radius of less than 16 follows, arranged according to radius and perimeter. Many obvious and unexpected relationships can be noted in this tabulation.

### SOLUTIONS OF PYTHAGOREAN TRIANGLES

**ARRANGED ACCORDING TO RADIi OF THE INSCRIBED CIRCLES**

<table>
<thead>
<tr>
<th>Radius</th>
<th>Generators</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Perimeter</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2, 1 (p)</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>3, 2 (p)</td>
<td>5</td>
<td>10</td>
<td>11</td>
<td>26</td>
</tr>
<tr>
<td>3</td>
<td>4, 3 (p)</td>
<td>7</td>
<td>20</td>
<td>21</td>
<td>48</td>
</tr>
<tr>
<td>4</td>
<td>5, 4 (p)</td>
<td>9</td>
<td>34</td>
<td>35</td>
<td>76</td>
</tr>
<tr>
<td>5</td>
<td>6, 5 (p)</td>
<td>2</td>
<td>50</td>
<td>51</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>6, 1 (p)</td>
<td>22</td>
<td>10</td>
<td>31</td>
<td>70</td>
</tr>
<tr>
<td>2\sqrt{5}, \sqrt{7}</td>
<td>13</td>
<td>18</td>
<td>21</td>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>
### THE MAIL BAG

Sons! Sons! Sons! Everybody is having boy babies. If we don't be getting some duodecimal daughters pretty soon, what are we going to do about double-duodecimal grandchildren?

Mr. and Mrs. Janison Handy, Jr., announce the debut of Galen William Handy. Two of our Canadian families, Mr. and Mrs. Leon L'Heureux, and Mr. and Mrs. Edwin Bobyn jubilate over sons. It is the second son for the L'Heureuses. Paul and Cam Adams (the Mad Adamses) have a son. Bob and Mary Lloyd have their third son. Ain't nobody got a daughter? Something will have to be done about this!

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We are delighted to present Trenchard More's article on An Exponential Expression for Music. This scholarly paper supplements beautifully the earlier work of Velizar Godjevatz on the New Musical Notation which appeared in the Bulletin for October, 1948.

Not only is Trenchard More to be congratulated on a fine piece of work, but we are to be congratulated on having among the rising generation so capable an advocate of duodecimals. We expect to hear the reverberations of the note these pioneers have struck.

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We are eager to find among our members a source for a paper on a duodecimal color notation. The Munsell decimal color notation is important commercially, but it suffers from some distortion in being compressed within the decimal limitation. A duodecimal notation would be a definite refinement, and would require but minor modification of the Munsell System.

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Louis Paul d'Autremont's article on the Duodecimal Perpetual Calendar has had a fine reception. In the Havana daily Informacion, Juan de Dios Tejada commented favorably and at considerable length on the d'Autremont proposal in his column, La Marcha de la Tecnica.

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The Board of Directors of the Society has recently authorized public announcement of our willingness to furnish without charge sets of introductory duodecimal literature to the pupils of mathematics classes of teachers colleges, to the extent that our supply permits. Sets will comprise a copy of the Duodecimal Bulletin, a copy of the reprint of the Excursion in Numbers, by F. Emerson Andrews, and a copy of the Society's folder.

Ye Ed.

### EXTRA COPIES

Many of those interested in duodecimals sedulously maintain complete files of the Duodecimal Bulletin. This is a practice which we wish to encourage. These esteemed people are sometimes confronted with the need for cutting out some table or article from the Bulletin for special use. On request, we will gladly supply extra copies, so that they can clip the needed material, yet maintain their Bulletin files in proper condition.